§2. Dirac and the Spinor

§2.1 Representations of the Lorentz group

Einstein's theory of relativity states that different "inertial frames" with coordinates x'^μ and x^μ are equivalent iff

$$\eta_{\mu\nu}\, dx'^\mu dx'^\nu = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

where $\eta_{\mu\nu}$ is the diagonal matrix

$\eta_{00} = 1, \quad \eta_{ii} = -1 \quad , i=1,2,3$

$$\longrightarrow \qquad \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = \eta_{\rho\sigma} \qquad (1)$$

$\longrightarrow$ linear trf. $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

with $\quad \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad (2)$

$\longrightarrow$ induce unitary linear transformations on states in physical Hilbert space

$$\Psi \longmapsto U(\Lambda, a)\Psi$$

with $\quad U(\Lambda', a')\, U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a') \qquad (3)$

"inhomogeneous Lorentz group" or

"Poincaré group"

Setting $a=0$ $\rightarrow$ "homogeneous Lorentz group"
infinitesimal transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad a^\mu = \varepsilon^\mu$$

where $\omega^\mu{}_\nu, \varepsilon^\mu \ll 1$

$\rightarrow$ equation (2) reads

$$\eta_{\rho\sigma} = \eta_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma)$$
$$= \eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2)$$

$$\rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}$$

an anti-symmetric second rank tensor in 4d has $(4\times 3)/2 = 6$ independent components. We have

$$U(\mathbb{1}+\omega,\varepsilon) = \mathbb{1} + \tfrac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} - i\varepsilon P^\rho + \dots$$

higher order terms

$$\rightarrow J^{\rho\sigma\dagger} = J^{\rho\sigma}, \quad P^{\rho\dagger} = P^\rho \text{ (unitarity)}$$
$$J^{\rho\sigma} = -J^{\sigma\rho} \quad (\omega \text{ is anti-sym})$$

As we will see: P^1, P^2, and P^3 are the components of the momentum operators, J^{23}, J^{31}, and J^{12} are components of the angular momentum, and P^0 is the Hamiltonian

Consider the product

$$U(\Lambda, a)\, U(\mathbb{1}+\omega, \varepsilon)\, U^{-1}(\Lambda, a)$$

unrelated to ω, ε

$$(3) \longrightarrow \underbrace{U(\Lambda^{-1}, -\Lambda^{-1}a)}\, U(\Lambda, a) = U(\mathbb{1}, 0)$$

$$\longrightarrow \quad = U^{-1}(\Lambda, a)$$

$$\longrightarrow U(\Lambda, a)\, U(\mathbb{1}+\omega, \varepsilon)\, U^{-1}(\Lambda, a)$$

$$= U(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}, \Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a)$$

To first order in $\omega_{\rho\sigma}$ and ε_ρ we have then

$$U(\Lambda, a)\left[\tfrac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma} - \varepsilon_\rho P^\rho\right]U^{-1}(\Lambda, a)$$

$$= \tfrac{1}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu} J^{\mu\nu} - (\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

equating coefficients of $\omega_{\rho\sigma}$ and ε_ρ on both sides, we find

$$U(\Lambda, a)\, J^{\rho\sigma}\, U^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\rho (J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu)$$

$$U(\Lambda, a)\, P^\rho\, U^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho P^\mu \qquad (4)$$

$\rightarrow$ under homogeneous Lorentz trfs. $(a=0)$, P^μ is a vector and $J^{\mu\nu}$ is a tensor

Setting now $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $a^\mu = \varepsilon^\mu$, $\omega, \varepsilon \ll 1$, with ω, ε unrelated to previous ω, ε

$$(4) \rightarrow \; i\left[\tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, J^{\rho\sigma}\right]$$
$$= \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} - \varepsilon^\rho P^\sigma + \varepsilon^\sigma P^\rho$$
$$i\left[\tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, P^\rho\right] = \omega_\mu{}^\rho P^\mu$$

equating coefficients of $\omega_{\mu\nu}, \varepsilon_\mu$ on both sides

$$\rightarrow i\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}$$
$$i\left[P^\mu, J^{\rho\sigma}\right] = \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho \qquad (5)$$
$$\left[P^\mu, P^\rho\right] = 0$$

$\rightarrow$ Lie algebra of Poincaré group

Operators that commute with energy $H = P^0$:

$$\vec{P} = \{P^1, P^2, P^3\}$$
$$\vec{J} = \{J^{23}, J^{31}, J^{12}\}$$

and P^0 itself. Remaining generators form "boost" three-vector:

$$\vec{K} = \{J^{01}, J^{02}, J^{03}\}$$

$\rightarrow$ <u>not</u> conserved! Hence physical states are <u>not</u> labeled by eigenvalues of $\vec{K}$

In three-dimensional notation:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \qquad i,j,k = 1,2,3$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k, \qquad \varepsilon_{123} = +1$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k,$$
$$[J_i, P_j] = i\varepsilon_{ijk} P_k,$$
$$[K_i, P_j] = -i H \delta_{ij},$$
$$[J_i, H] = [P_i, H] = [H, H] = 0$$
$$[K_i, H] = -i P_i$$

Looking at the homogeneous Lorentz group, i.e. $\{J_i, K_i\}$, we can form combinations

$$J_{\pm i} = \frac{1}{2}(J_i \pm i K_i)$$

$$\rightarrow \quad [J_{+i}, J_{+j}] = i\varepsilon_{ijk} J_{+k} \qquad su(2)\text{-alg}$$
$$[J_{-i}, J_{-j}] = i\varepsilon_{ijk} J_{-k} \qquad su(2)\text{-alg}$$
$$[J_{+i}, J_{-j}] = 0$$

mathematically: $su(2) \otimes su(2) \cong so(1,3)$

$\rightarrow$ representations of $so(1,3)$ can be labeled as

$(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(1,0)$, $(0,1)$, $(\frac{1}{2},\frac{1}{2})$, ...

§2.2 The Dirac Equation

We know the Klein-Gordon equation:

$$(\partial^2 + m^2)\varphi = 0$$

$\rightarrow$ relativistic wave equation quadratic in spacetime derivatives, describing a free scalar particle of mass m

Question: Is there a relativistic (Dirac, 1928) wave equation linear in spacetime derivatives $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$?

$$\rightarrow \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad (1)$$

γ^μ cannot simply be 4 numbers (not Lorentz invariant)

$\rightarrow$ acting with $(i\gamma^\mu\partial_\mu + m)$ gives:

$$-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = 0$$

Now define "anti-commutator"

$$\{A, B\} = AB + BA$$

$$\rightarrow \quad \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu$$

and therefore,

$$\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

Thus
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad (2)$$
with $\eta^{\mu\nu}$ the Minkowski metric, giving
$$(\partial^2 + m^2)\psi = 0$$
Now
$$\eta = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$\longrightarrow \quad (\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$

Last condition implies that the γ^μ cannot be ordinary numbers!

Clifford algebra

A set of objects satisfying condition (2) is called a "Clifford algebra"

One solution of equations (2) is given by the followin 4x4 matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \mathbb{1} \otimes \sigma^3, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \sigma^i \otimes i\sigma^2$$

"gamma matrices"

where the σ^i, $i=1,2,3$ are the standard Pauli matrices

We also define $\gamma_\mu := \eta_{\mu\nu}\gamma^\nu$

Let us compute:

$$\begin{aligned}\gamma^i\gamma^j &= (\sigma^i \otimes i\sigma^2)(\sigma^j \otimes i\sigma^2)\\ &= (\sigma^i\sigma^j \otimes i^2\sigma^2\sigma^2)\\ &= -(\sigma^i\sigma^j \otimes \mathbb{1})\end{aligned}$$

$$\longrightarrow \{\gamma^i,\gamma^j\} = -\{\sigma^i,\sigma^j\}\otimes \mathbb{1} = -2\delta^{ij} \checkmark$$

<u>Note</u>: There is no representation of the Clifford algebra in 4d with the γ^μ smaller than 4×4 matrices!

Next, let's transform to momentum space:

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\,\psi(p)$$

$\longrightarrow$ plugging into (1) gives:

$$(\gamma^\mu p_\mu - m)\psi(p) = 0 \qquad (3)$$

Lorentz invariance:

We now want to show that (3) is Lorentz invariant. Define

$$\mathcal{J}^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$\rightarrow$ $[\mathcal{J}^{\mu\nu}, \gamma^\lambda] = i(\gamma^\mu \eta^{\nu\lambda} - \gamma^\nu \eta^{\mu\lambda})$

$\rightarrow$ $\mathcal{J}^{\mu\nu}$ satisfies Lorentz algebra (§2.1, eq. (5))

Let $\psi(p)$ be in representation

$$U(\Lambda)\psi(p) = e^{-\frac{1}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\psi(p)$$

Using $\mathcal{J}^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$, gives for $\psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$ under angular momentum transformations: $\phi \mapsto e^{-i\omega_{12}\frac{1}{2}\sigma^3}$,

$$\chi \mapsto e^{-i\omega_{12}\frac{1}{2}\sigma^3}$$

$\rightarrow$ ψ is in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ rep. of Lorentz group

Moreover,

$$(i\gamma^\mu p'_\mu - m)\psi' = (i\gamma^\mu \Lambda_\mu{}^\nu p_\nu - m)U(\Lambda)\psi$$

$$= (i\, U(\Lambda)\gamma^\mu U(\Lambda)^{-1} p_\mu - m)U(\Lambda)\psi$$

$$= U(\Lambda)(i\gamma^\mu p_\mu - m)\psi = 0$$

where we used

infinetisam $\omega_{\mu\nu}$

$$U(\Lambda)\gamma^\lambda U(\Lambda)^{-1} = \gamma^\lambda - \frac{i}{2}\omega_{\mu\nu}[\gamma^{\mu\nu}, \gamma^\lambda]$$

$$= \gamma^\lambda + \gamma^\mu \omega_\mu{}^\lambda$$

or in general $= \gamma^\mu \Lambda_\mu{}^\lambda$

i.e. γ^μ is a "Lorentz vector"

$\square$